# Pointing Geometry for Low Earth Orbit Auroral Observations

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## Introduction

ALGORITHMS are developed to solve the geometry required to point telescopes mounted on a low Earth-orbiting satellite at features in the aurora (at height of typically 100 km above Earth's surface). It is based on requirements for the proposed AURIO telescopes, which may be flown on a polar platform of the European Space Agency (ESA). The idea of pointing an instrument and tracking a target from space is not new and the geometry has been solved for several cases (see, e.g., Jerkovsky,[1] Burdick et al.,[2] and Hablani[3]). The traditional approach is to use vector notation, but at the implementation stage this method usually needs to be supported by a library of subroutines to implement the basic vector operations of addition, cross and scalar products, and rotations of vectors about the coordinate axes, which raises difficulties in a small embedded computer system.

In this Note a computational method is presented to find the line-of-sight (LOS) vector in the satellite-body-attached coordinate frame. This method uses homogeneous transformation matrices (normally used in robot kinematics) to provide efficient algorithms that can readily be implemented on-board the satellite. A spherical Earth is assumed in this work as the basic model. However, the spheroidal model of Earth has been used to assess errors due to this assumption. The software coding of the pointing equations can easily be implemented in any high level programming language, and they have been tested in the occam language.

## Coordinate Frames

To solve the pointing geometry problems, three coordinate frames are used. These are first the master coordinate frame (MCF) *OXYZ*, which is a geocentric coordinate system suited to satellite navigation and commanding pointable instruments on the spacecraft. This is a right-handed coordinate system with its origin at the center of Earth, the $Z$ axis lies along Earth's spin axis and the $X$ axis coincides with the vernal equinox. This master frame of reference is locally inertial and does not rotate with Earth. A second coordinate frame, $Sxyz$, is defined for the attitude of the spacecraft and is named the satellite coordinate frame (SCF). The origin $S$ is the satellite center of mass, which is the same as the origin for the mechanical axes of the spacecraft (see later). The $z$ axis is pointing in the direction of the line from $S$ to the center of Earth, the $y$ axis is orthogonal to the orbit plane, and the $x$ axis completes a right-handed coordinate system. The SCF, $Sxyz$, is related to the $MCF$, $OXYZ$, by the orbital plane specification $(\Omega, i)$ and the spacecraft position in the orbit $(\phi, r)$. The final coordinate system, $Suvw$, is the local coordinate frame (LCF), which is defined by the mechanical axes of the spacecraft. The origin of this frame is the same as for the SCF.

The LFC is related to the SFC by the attitude information (IA); that is, the SCF can be converted to coincide with the LCF by some rotations about the $x$, $y$, and $z$ axes, i.e., roll, pitch, and yaw, respectively. It is in this frame (the LCF) that the command angles for pointing an instrument must be calculated (assuming that the small displacement of the pointing instruments from the center of mass of the satellite may be ignored).

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## Spacecraft Position in Orbital Plane

The spacecraft position in the orbital plane is assumed to be given in the form of $\phi$ and $r$, where $\phi$ is the angle at the center of Earth between the line of nodes and the position vector of the satellite and $r$ is the length of this vector. These can be derived (see, e.g., Wertz[4]) from the six classical orbital elements: right ascension of ascending node $(\Omega_0)$, inclination $(i_0)$, argument of perigee $(\omega_0)$, semimajor axis of the ellipse $(a_0)$, eccentricity $(e_0)$, and mean anomaly $(M_0)$. The target coordinates used in this work for the calculation of pointing direction of the pointable instrument are taken to be given in the MCF $(A_X, A_Y, A_Z)$; see Fig. 1.

## Target Pointing and Tracking

As a result of defining the different modes of operation for the AURIO[5] telescopes, three pointing and tracking algorithms have been introduced as follows: 1) a target-pointing algorithm for use with a specified target, 2) a real-time algorithm that changes the pointing direction in response to the real-time commands initiated from the ground station by increasing or decreasing the command angles $(\theta_A, \theta_E)$, and 3) an automatic tracking algorithm that calculates the target coordinates at any instant, given the current direction of the pointable instrument (LOS) and the target altitude or its range from the satellite. Here we are mainly concerned with the first of these tasks.

As is shown in Fig. 1, the pointing direction can be specified in terms of two angles in the spacecraft body-attached coordinate frame $(\theta_A, \theta_E)$, given the spacecraft and target positions. As an initial simplification it is assumed that the spacecraft has the nominal attitude. Now the problem is finding the target coordinates in SCF $(A_x, A_y, A_z)$, and then

$$\theta_A = \arctan \frac{A_x}{A_z} \tag{1}$$

$$\theta_E = \arctan \frac{-A_y}{\sqrt{A_x^2 + A_z^2}} \tag{2}$$

The problem is thus reduced to finding a transformation matrix $T$ that relates the SCF to the MCF:

$$(A_X, A_Y, A_Z, 1)^T = T(A_x, A_y, A_z, 1)^T \tag{3}$$

The introduction of a fourth coordinate or component to specify a vector in a three-dimensional space represents a Cartesian vector in homogeneous coordinates. The homogeneous transformation matrix is a 4 × 4 matrix that maps a position vector expressed in homogeneous coordinates from one coordinate system to another

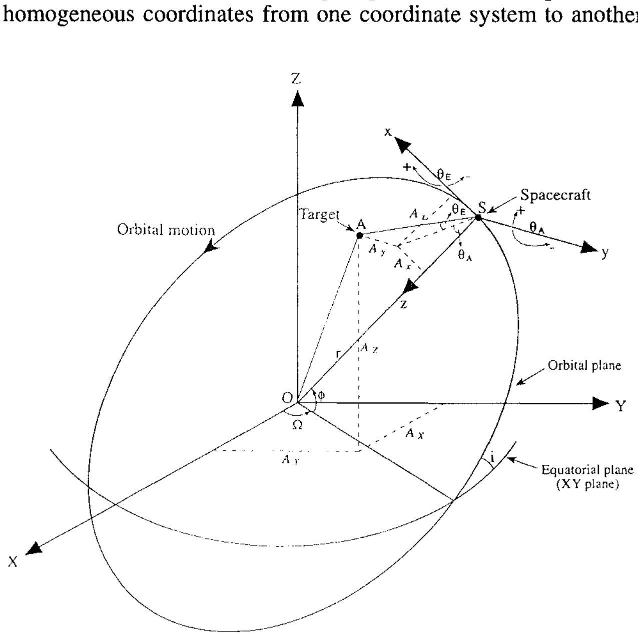


**Fig. 1 Implication of target-pointing problem; command angles $(\theta_A, \theta_E)$ are specified in satellite coordinate frame (Sxyz).**

coordinate system. There exist six standard basic homogeneous transformation matrices that can be multiplied together to obtain a composite homogeneous transformation matrix.[6]

To determine the composite homogeneous transformation matrix $T$ to map the target position vector from the MCF to the SCF, the following rotations and translations are required (see Fig. 2): 1) a rotation of $\Omega$ angle about the $Oz$ axis, 2) a rotation of $-(90 - i)$ angle about the rotated $Ox$ axis $(Ox_1)$, 3) a rotation of $-(90 + \phi)$ angle about the rotated $Oy$ axis $(Oy_2)$, and 4) a translation of $-r$ units along the rotated $Oz$ axis $(Oz_3)$. It is assumed that, initially, both coordinate frames, $OXYZ$ and $Oxyz$, are coincident; then $Oxyz$ has some rotations and translations about its own principal axes until, finally, it coincides with the SCF $(Sxyz)$. The composite homogeneous transformation matrix $T$, which represents the above transformation, can be found by premultiplication of four standard homogeneous matrices representing these rotations/translations:

$$T = T_{Oz,\Omega} T_{Ox,-(90-i)} T_{Oy,-(90+\phi)} T_{Oz,-r}$$

If $T$ is presented in the form

$$T = \begin{vmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

then the vectors $n$, $s$, $a$, and $p$ are found to be

$$n = \begin{vmatrix} n_x \\ n_y \\ n_z \end{vmatrix} = \begin{vmatrix} -\cos(\Omega)\sin(\phi) - \sin(\Omega)\cos(i)\cos(\phi) \\ -\sin(\Omega)\sin(\phi) + \cos(\Omega)\cos(i)\cos(\phi) \\ \sin(i)\cos(\phi) \end{vmatrix} \tag{4}$$

$$s = \begin{vmatrix} s_x \\ s_y \\ s_z \end{vmatrix} = \begin{vmatrix} -\sin(\Omega)\sin(i) \\ \cos(\Omega)\sin(i) \\ -\cos(i) \end{vmatrix} \tag{5}$$

$$a = \begin{vmatrix} a_x \\ a_y \\ a_z \end{vmatrix} = \begin{vmatrix} -\cos(\Omega)\cos(\phi) + \sin(\Omega)\cos(i)\sin(\phi) \\ -\sin(\Omega)\cos(\phi) - \cos(\Omega)\cos(i)\sin(\phi) \\ -\sin(i)\sin(\phi) \end{vmatrix} \tag{6}$$

$$p = \begin{vmatrix} p_x \\ p_y \\ p_z \end{vmatrix} = r \cdot \begin{vmatrix} \cos(\Omega)\cos(\phi) - \sin(\Omega)\cos(i)\sin(\phi) \\ \sin(\Omega)\cos(\phi) + \cos(\Omega)\cos(i)\sin(\phi) \\ \sin(i)\sin(\phi) \end{vmatrix} \tag{7}$$

The target coordinates are to be given in the MCF, but it is required in the SCF. Therefore, the problem is reverse conversion from coordinates in the MCF $(A_X, A_Y, A_Z)$ to coordinates in the SCF $(A_x, A_y, A_z)$:

$$(A_x, A_y, A_z, 1)^T = T^{-1}(A_X, A_Y, A_Z, 1)^T$$

In general, the inverse of $T(T^{-1})$ can be found to be[6]

$$T^{-1} = \begin{vmatrix} n_x & n_y & n_z & b_x \\ s_x & s_y & s_z & b_y \\ a_x & a_y & a_z & b_z \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

where

$$b = \begin{vmatrix} b_x \\ b_y \\ b_z \end{vmatrix} = \begin{vmatrix} -(n_x p_x + n_y p_y + n_z p_z) \\ -(s_x p_x + s_y p_y + s_z p_z) \\ -(a_x p_x + a_y p_y + a_z p_z) \end{vmatrix} \tag{8}$$

Thus, the target coordinates in the SCF $(A_x, A_y, A_z)$ can be derived from its coordinates in the MCF via the following equations:

$$A_x = n_x A_X + n_y A_Y + n_z A_Z + b_x \tag{9}$$

$$A_y = s_x A_X + s_y A_Y + s_z A_Z + b_y \tag{10}$$

$$A_z = a_x A_X + a_y A_Y + a_z A_Z + b_z \tag{11}$$

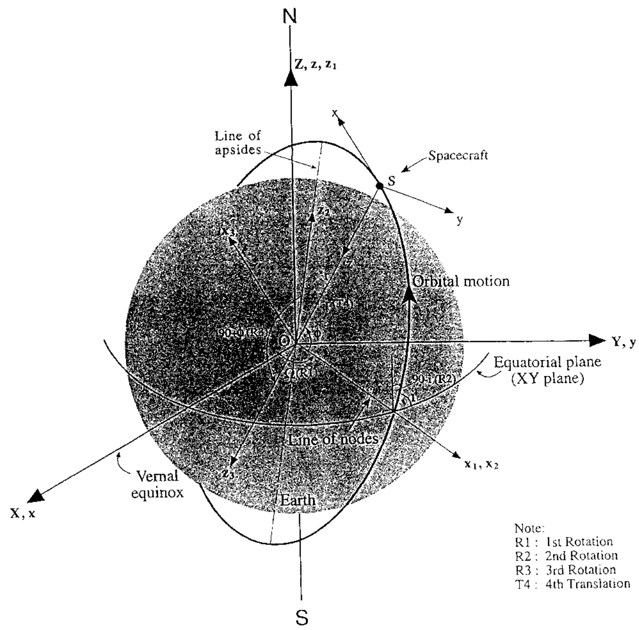


**Fig. 2 Transformation from master coordinate frame (*OXYZ*) to satellite coordinate frame (*Sxyz*).**

where the elements of the vectors $n$, $s$, $a$, and $b$ can be substituted from Eqs. (5–8), respectively. Then $\theta_A$ and $\theta_E$ (see Fig. 1) can be calculated in the SCF from Eqs. (1) and (2).

As mentioned earlier, the above equations for deriving the command angles are applicable when the spacecraft is in a stabilized attitude with zero or negligible attitude errors, i.e., roll, pitch, and yaw equal to zero. Any changes in the spacecraft attitude can be considered as rotations about the $Sx$, $Sy$, and $Sz$ axes in sequence. If the roll, pitch, and yaw angles are called $\alpha$, $\beta$, and $\gamma$, then the composite homogeneous transformation matrix can be expanded in the general form

$$T_G = T T_{Sx,\alpha} T_{Sy,\beta} T_{Sz,\gamma} = T T_A$$

where $T_A$ is the transformation matrix for attitude.

Then the target coordinates and command angles can be derived in the spacecraft local frame of reference $(Suvw)$:

$$(A_u, A_v, A_w, 1)^T = T_G^{-1}(A_X, A_Y, A_Z, 1)^T \tag{12}$$

$$\theta_A = \arctan \frac{A_u}{A_w} \tag{13}$$

$$\theta_E = \arctan \frac{-A_v}{\sqrt{A_u^2 + A_w^2}} \tag{14}$$

Nevertheless, the spacecraft that will carry the AURIO complex of instruments is assumed to be, in its nominal operation, in the yaw steering mode such that the spacecraft is slowly oscillated about its yaw axis (nadir direction) to compensate exactly for apparent sideways motion due to the rotation of Earth beneath the spacecraft; see Ref. 7 for details. Thus, the normal in the attitude of the spacecraft is the rotation about $Sz$ or yaw axis, and the general transformation matrix $T_G$ can be found to be

$$T_G = T T_{Sz,\gamma} = T \begin{vmatrix} \cos(\gamma) & -\sin(\gamma) & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Then the target coordinates and command angles can be derived, in $Suvw$ coordinate frame, from Eqs. (12–14).

## Conclusions

Homogeneous transformation matrices were used to solve the general problem of pointing a two-degree-of-freedom instrument at a target. These lead to equations that may be readily implemented in an on-board control algorithm. Using these methods in solving the geometry problem in this work makes it easy to understand the transitions between the different coordinate frames and to implement in a small on-board computer. Straightforward extensions to this work have developed algorithms for single-degree-of-freedom pointing systems (e.g., when the target can be taken to be in the orbital plane) and to invert the problem to derive the target location needed for automatic tracking.

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# Low-Earth-Orbit Maintenance: Reboost vs Thrust–Drag Cancellation

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### Introduction

WE define the problem of orbit maintenance within an atmosphere as keeping the spacecraft within a specified altitude band about a mean circular orbit. One interesting solution to problem is thrust–drag cancellation,

$$T - D = 0 \tag{1}$$

and thrust vectoring along the velocity vector,

$$\frac{\mathbf{T}}{T} = \frac{\mathbf{v}}{v} \tag{2}$$

resulting in a forced Keplerian trajectory (FKT). Although the control law $T = D$ is quite difficult to achieve physically because of uncertainties in drag modeling (atmospheric density and ballistic coefficient) and thruster designs (on–off), it is typically used to determine the fuel budget required for orbit maintenance.[1] A more practical solution to the orbit maintenance problem is to periodically reboost the spacecraft. Nonetheless, we investigate the fuel optimality of an FKT by considering the totality of extremal arcs. Barring the special case when $T_{\max} = D$, an optimal FKT must necessarily be a singular arc.

It can be shown[2,3] that, when both the thrust magnitude and its direction are control parameters, an FKT is not a Mayer-extremal arc and hence not fuel optimal, i.e.,

$$T^* \neq D, \qquad \frac{\mathbf{T}^*}{T^*} \neq \frac{\mathbf{v}}{v} \tag{3}$$

where the asterisk denotes the optimal values. Unfortunately, this analysis breaks down when Eq. (2) is imposed as a constraint since, although the derivation of the optimal steering is decoupled from that of the thrust magnitude, the converse is not true. Thus, the question remains whether the control law of Eq. (1) is optimal under the steering constraint imposed by Eq. (2): Is $T^* = D$ when $\mathbf{T}/T = \mathbf{v}/v$? Although this question was addressed in Ref. 4 for the special case of a "forced circular orbit," our approach and motivation are quite different in the sense that we seek not only the answer to the more general case of a Keplerian arc but also the ramifications of the extremal solution $T^*$. To this end, we derive the extremal singular thrust arc $T_s^*$ in state variable feedback form and demonstrate some interesting consequences. In addition, by way of a linear analysis, we show heuristically that the difference in propellant consumption between an FKT and periodic Hohmann transfers is zero (i.e., no greater than the order of the approximations). The following sections elaborate the details of these findings.

### Extremal Arcs

The objective of this section is to determine the extremal arcs of a time-free, Mayer-optimal control problem of transferring a spacecraft from some initial manifold to a terminal manifold while minimizing a generic performance index,

$$J = y(r_f, v_f, \gamma_f, m_f) \tag{4}$$

where $f$ denotes the final values and $r$, $v$, $\gamma$, and $m$ are the variables corresponding to the radial position, speed, flight-path angle, and mass, respectively. The equations of motion for coplanar flight of an endo-atmospheric low-earth-orbit (LEO) spacecraft are

$$\begin{bmatrix} \dot{r} \\ \dot{v} \\ \dot{\gamma} \\ \dot{m} \end{bmatrix} = \underbrace{\begin{bmatrix} v \sin\gamma \\ \dfrac{-D}{m} - g\sin\gamma \\ \left(\dfrac{v^2}{r} - g\right)\dfrac{\cos\gamma}{v} \\ 0 \end{bmatrix}}_{a_0} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{m} \\ 0 \\ \sigma \end{bmatrix}}_{a_1} T \tag{5}$$

where the significance of $a_0$ and $a_1$ will be apparent later. Here, $T$ is the thrust, $D$ the atmospheric drag, $g$ the gravitational acceleration, and $\sigma$ the negative inverse of the exhaust speed. These parameters are modeled as

$$D = \tfrac{1}{2}\rho(r)v^2 A C_D, \qquad g = \mu/r^2, \qquad \sigma = -1/g_0 I_{sp} \tag{6}$$

where $\rho(r)$ is a spherically symmetric atmospheric density, $A$ the spacecraft's reference area, $C_D$ the drag coefficient, $\mu$ the gravitational constant, $I_{sp}$ the specific impulse, and $g_0$ the gravitational acceleration at some reference altitude (sea level).

The Pontryagin $H$-function[5] for this problem is given by

$$H = \lambda_r v \sin\gamma + \lambda_v\left(\frac{T-D}{m} - g\sin\gamma\right) + \lambda_\gamma\left(\frac{v^2}{r} - g\right)\frac{\cos\gamma}{v} + \lambda_m \sigma T \tag{7}$$

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